

Quantum Mechanics as a Classical Theory XVI: Positive-Definite Phase-Space Densities

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Abstract

In this paper we will turn our attention to the problem of obtaining phase-space probability density functions. We will show that it is possible to obtain functions which assume only positive values over all its domain of definition.

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1 Introduction

Since the very moment, when the phase-space probability density representation was introduced to reproduce, in a manner as close as possible to classical statistical mechanics, the results of the expectation values predicted by quantum mechanics[1], there have appeared the discomfort of being capable of making this description only appealing to probability densities having negative values within some range of its phase-space arguments (x, p) .

Many papers have appeared since then trying to push this representation method to its utmost formal developments[2]. The negativity problem was considered by some as deriving from the non-commutativity character of operators corresponding to canonical conjugate functions, or just representing the fact that quantum mechanics, being an essentially non-classical theory, wouldn't be adjustable to a phase-space description. Others have thought this problem as irrelevant, for the phase-space distribution obtained (by Wigner-Moyal-Weyl procedure, for example) will furnish all the relevant expectation values identical with those of usual quantum mechanics. Indeed it will give even more, since expectations related with products of the type $x^n p^m$ will be unambiguous calculated within this formulation of the problem—something the usual approach may do only after postulating some specific operator construction method.

Others, however, have seen the far reaching consequences of being capable to construct such a representation, without the inconveniences introduced by the negativity problem[3, 4, 5].

We will join this last class of authors on their concerns. We have been arguing, since the first paper of this series[6], that quantum mechanics is just a classical statistical theory, defined upon configuration space, related with the thermodynamic equilibrium situations of physical systems; this means that it is more restricted than the theory related with Liouville's equation, which is classical statistical mechanics *strictu sensu*. However, the element of liaison between the Liouville classical statistical mechanics, defined upon phase-space, and the Schrödinger quantum mechanics upon configuration-space was, until presently, the phase-space probability density function assuming negative values. If this function does assume negative values within some ranges of x and p values, then no acceptable physical interpretation may be appended to it. Then comes the discomfort of having all the theory based upon an element of irrationality (unphysical).

The task of being capable to obtain positive definite probability densities will thus fulfill this epistemological void of the theory, giving to it its final contours. This is the intention of the present paper.

In the second section we will show the method of deriving quantum mechanics, we had thus far adopted, from classical statistical mechanics[6]. While doing this, we will show what were our assumptions and why they did lead to the negativity problem. This critic development of the formalism will give us insight about the ways we may possibly overcome the difficulties.

In the third section we will use the insights furnished by the previous section to derive a method that gives us only positive definite phase-space probability density functions.

The results of the third section will be used to derive, as a naive application, the Ehrenfest's theorem, giving the dynamical behavior of the position and momentum expectations. This will be done in the fourth section.

In the fifth section we will compare the two procedures of finding phase-space probability density functions. This comparison will be made based upon the expectation values these two approaches furnish to the specific example of the harmonic oscillator.

The last section will be devoted to our conclusions.

In the appendix we make a comment about an interesting result, closely related with our own, made in the literature.

2 Previous Approach

In our previous approach to the problem, we began with the phase-space joint probability density function $F(x, p; t)$ defined as a convolution of two phase-space probability amplitudes $\phi(x, p; t)$

$$F(x, p; t) = \int_{-\infty}^{+\infty} \phi^\dagger(x, 2p - p'; t) \phi(x, p'; t) dp'. \quad (1)$$

We then defined the characteristic function $Z_Q(x, \delta x; t)$ as the *infinitesimal* Fourier transform

$$Z_Q(x, \delta x; t) = \int e^{ip\delta x/\hbar} F(x, p; t) dp, \quad (2)$$

and applied the above defined Fourier transform to Liouville's equation, which we assumed the function $F(x, p; t)$ satisfies, to find the equation satisfied by Z_Q as

$$-i\hbar \frac{\partial Z_Q(x, \delta x; t)}{\partial t} - \frac{\hbar^2}{m} \frac{\partial^2 Z_Q(x, \delta x; t)}{\partial x \partial (\delta x)} + \frac{\partial V(x)}{\partial x} \delta x Z_Q(x, \delta x; t) = 0. \quad (3)$$

Because of expressions (1) and 2), the characteristic function was given by

$$Z_Q(x, \delta x; t) = \psi^\dagger(x - \delta x/2; t) \psi(x + \delta x/2; t) \quad (4)$$

where we wrote, for the *complex* amplitudes,

$$\psi(x; t) = R(x; t) e^{iS(x; t)/\hbar}, \quad (5)$$

with $R(x; t)$ and $S(x; t)$ real functions. The characteristic function became, in terms of the $R(x; t)$ and $S(x; t)$ functions

$$Z_Q(x, \delta x; t) = \left\{ R(x; t)^2 + \left(\frac{\delta x}{2} \right)^2 \times \right.$$

$$\times \left[R(x;t) \frac{\partial^2 R(x;t)}{\partial x^2} - \left(\frac{\partial R(x;t)}{\partial x} \right)^2 \right] \exp \left(\frac{i}{\hbar} \delta x \frac{\partial S(x;t)}{\partial x} \right), \quad (6)$$

up to second order in the variable δx , since it is assumed as infinitesimal and equation (3) has to be considered up to order one on this variable.

Substituting these expressions in equation (3) we got the two equations

$$\frac{\partial R(x;t)^2}{\partial t} + \frac{\partial}{\partial x} \left[\frac{R(x;t)^2}{m} \frac{\partial S(x;t)}{\partial x} \right] = 0 \quad (7)$$

and

$$\frac{-i\delta x}{\hbar} \frac{\partial}{\partial x} \left\{ \frac{\partial S(x;t)}{\partial t} + \frac{1}{2m} \left(\frac{\partial S(x;t)}{\partial x} \right)^2 + V(x) - \frac{\hbar^2}{2mR(x;t)} \frac{\partial^2 R(x;t)}{\partial x^2} \right\} = 0, \quad (8)$$

where we used the independence between x and δx , and also the infinitesimal character of δx to retain only up to power one on this variable.

These two equations are mathematically equivalent to the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x;t)}{\partial x^2} + V(x)\psi(x;t) = i\hbar \frac{\partial \psi(x;t)}{\partial t}, \quad (9)$$

when we substitute the expression (5) on it, and separate the real and imaginary parts.

This process allowed us to find, upon inversion of the Fourier transformation (2), a phase-space probability density which will simulate all the results of ordinary quantum mechanics (the statistical moments, such as expectations, square deviations, etc..). However, by the very definition of the phase-space probability density $F(x, p; t)$ given in (1), we shall not expect to find positive-definite functions, since the phase-space amplitudes may oscillate—mainly for the excited states, where their corresponding configuration-space amplitudes show such an oscillatory behavior—and their product is defined at different momentum-space points in (1), as required for a convolution.

The process of operator formation was then simply introduced. Since Z_Q is a characteristic function (in momentum space) of the phase-space probability density function, all statistical moments may be obtained as

$$\overline{(x^n p^m)} = \int \lim_{\delta x \rightarrow 0} x^n \left(-i\hbar \frac{\partial}{\partial(\delta x)} \right)^m Z(x, \delta x; t) dx, \quad (10)$$

where no ambiguity appears, since x and δx are independent variables. Using the decomposition (4) we may connect these expectation values, calculated with the characteristic function Z_Q , with those calculated using the probability amplitudes. This can be accomplished by means of the expression (10) above with (4) for the characteristic function, after performing an expansion of (4) with respect to the infinitesimal variable δx . In this case, the position and momentum operators become $\hat{x} = x$ and $\hat{p} = i\hbar \partial / \partial x$, and no longer commute.

We have argued that it was the process of inversion of the infinitesimal Fourier transformation (2) which introduced such negativity into our phase-space distributions. Now, we may say that it is mainly the imposition of having the characteristic function Z_Q written as a product that has introduced this negativity, since it forces the phase-space probability density to be written as a convolution in momentum space. We note, however, that, except for this unacceptable behavior—which has no physical interpretation—the phase-space densities thus derived was really adequate to the problem, for the reasons above mentioned. We just note that, when writing the characteristic function as the Fourier integral in (2) and assuming the decomposition in (4) we have also introduced an specific operator construction method[13], which will give us a procedure to evaluating expectation values of functions as $x^n p^m$. This means that, when changing some of these definitions (e.g. the convolution in (1)), the operator construction method may also vary, giving possibly new expectation values for these product. However, the expectation values of the ‘lateral functions’ x^n and p^m will always have to agree with those obtained by usual quantum mechanics techniques.

We will now present a method to overcome the negativity problem. In the next section we will present a new derivation of the problem in which we assume, from the very beginning the positive character of the phase-space probability density function.

3 Positive Densities

We begin by defining the phase-space probability density as

$$F(x, p; t) = \phi^\dagger(x, p; t)\phi(x, p; t), \quad (11)$$

where $\phi(x, p; t)$ are probability amplitudes defined upon phase-space. This function will thus be obviously positive definite.

The characteristic function continues to be defined by expression (2), but now we will also define the *characteristic amplitudes*

$$\xi(x', x; t) = \int e^{ipx'/\hbar} \phi(x, p; t) dp, \quad (12)$$

to write Z_Q , in terms of these functions, as

$$Z_Q(x, \delta x; t) = \int \xi^*(x', x; t) \xi(x' + \delta x, x; t) dx', \quad (13)$$

which now is a convolution. This was readily expected, since now we are defining the probability density $F(x, p; t)$ as a product upon phase-space, and $F(x, p; t)$ and $Z_Q(x, \delta x; t)$ are the Fourier transform of each other. We thus have fixed the positivity of the phase-space probability density function while letting the

characteristic function to possibly assume negative values, since this will not be physically unacceptable.

Another way of writing expression (13) is

$$Z_Q(x, \delta x; t) = \int \xi^*(x', x; t) e^{i\delta x \hat{p}' / \hbar} \xi(x', x; t) dx', \quad (14)$$

where

$$\hat{p}' = -i\hbar \frac{\partial}{\partial x'}. \quad (15)$$

We may substitute expression (14) into equation (3) to find the equation satisfied by the characteristic amplitudes ξ as

$$-i\hbar \frac{\partial \xi(x', x; t)}{\partial t} - \frac{\hbar^2}{m} \frac{\partial^2 \xi(x', x; t)}{\partial x \partial x'} + \frac{\partial V(x)}{\partial x} x' \xi(x', x; t) = 0, \quad (16)$$

which was expected, since the phase-space amplitudes also satisfy a Liouville equation[4] and the definition of the ξ 's parallels that of the characteristic function Z_Q .

It is obvious that all the statistical moments related to momentum or position values are still obtainable from the characteristic function Z_Q . In fact any function of the type $O[x, p]$ will have its expectation value, when calculated using the characteristic amplitudes, given by

$$\overline{O[x, p]} = \int \int \xi^*(x', x; t) \hat{O}[\hat{x}, \hat{p}'] \xi(x', x; t) dx' dx, \quad (17)$$

where

$$\hat{O}[\hat{x}, \hat{p}'] = O\left[x, -i\hbar \frac{\partial}{\partial x'}\right] \quad (18)$$

and no ambiguities appear, for x and x' are independent variables. These results come from the very application of the definitions (10) to the new definition (13). The explicit results are

$$\overline{(x^n p^m)} = \int \int \xi(x', x; t) x^n \left(-i\hbar \frac{\partial}{\partial x'}\right)^m \xi(x', x; t) dx' dx. \quad (19)$$

We may now put

$$\xi(x', x; t) = \psi(x'; t) \psi^\dagger(x; t) \quad (20)$$

to get, for the characteristic function

$$Z_Q(x, \delta x; t) = \psi(x; t) \psi^\dagger(x; t) \int \psi^\dagger(x'; t) \exp\left(\delta x \frac{\partial}{\partial x'}\right) \psi(x'; t) dx', \quad (21)$$

giving, for the lateral statistical moments

$$\overline{(p^m)} = \int \psi^\dagger(x'; t) \left(-i\hbar \frac{\partial}{\partial x'}\right)^m \psi(x'; t) dx' \quad (22)$$

and

$$\overline{(x^n)} = \int \psi^\dagger(x; t)(x^n)\psi(x; t)dx, \quad (23)$$

whereas, for the general product, we have

$$\overline{(x^n p^m)} = \int \psi^\dagger(x; t)(x^n)\psi(x; t)dx \int \psi^\dagger(x'; t) \left(-i\hbar \frac{\partial}{\partial x'}\right)^n \psi(x'; t)dx'. \quad (24)$$

Note that, now, the characteristic function became a quadrilinear function of the amplitudes[3]. This new definition of the characteristic function will lead to an operator construction method distinct from the one presented in the last section.

Until now we have not specified which equation the probability amplitudes $\psi(x; t)$ satisfy—indeed, this was done in the previous section but, since our definitions has changed, we are not certain if these amplitudes still obey the Schrödinger equation. However, if we turn our attention to the operator formation procedure given in (22) and (23), we thus see that the expectation of the energy function, given by

$$\overline{E} = \int \int \left(\frac{p^2}{2m} + V(x) \right) F(x, p; t) dx dp, \quad (25)$$

may be written, in terms of the probability amplitudes, as

$$\overline{E} = \int \psi^\dagger(x; t) \left[\frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x} \right)^2 + V(x) \right] \psi(x; t) dx, \quad (26)$$

which implies the Schrödinger equation

$$\frac{\hat{p}^2}{2m} \psi(x, t) + V(x)\psi(x; t) = E\psi(x; t), \quad (27)$$

with $\hat{p} = -i\hbar \partial / \partial x$. This suffices to prove that the amplitudes $\psi(x; t)$ still obey the Schrödinger equation.

The last expression for the expectation values of coordinate-momentum powers (24) will thus strongly differ from the one we have obtained in the last section. We also note that, in the present procedure, we still do not introduce any commutation problem, for x and x' continue to be independent variables, just like x and δx are. Since we showed that the probability amplitudes satisfy the Schrödinger equation, all the ‘lateral’ expectation values predicted by quantum mechanics will be reproduced by the present theory. The discrepancy between this section approach and the one described in the previous section may not be solved within the realm of the usual quantum formalism, for both approaches predict the same lateral statistical expectations (x^n or p^m alone), which are the only ones guaranteed by the above mentioned formalism—the

coordinate-momentum product expectations depending on the particular operator construction method (the correspondence rule) each approach proposes.

Using now the definition (14) and the decomposition (20), we have

$$\psi^\dagger(x; t)\psi(x'; t) = \int e^{ipx'/\hbar} \phi(x, p; t) dp, \quad (28)$$

which implies

$$\phi(x, p; t) = \psi^\dagger(x; t)\varphi(p; t), \quad (29)$$

where

$$\varphi(p; t) = \int e^{-ipx'/\hbar} \psi(x'; t) dx'. \quad (30)$$

With the result (29), for the phase-space probability amplitudes, and the initial imposition (11) we get

$$F_n(x, p; t) = |\psi_n(x; t)|^2 |\varphi_n(p; t)|^2, \quad (31)$$

as the desired positive definite phase-space probability density function. This result reinforces the F_n property of giving all the lateral statistical expectations as the ones predicted by usual quantum mechanics.

In the literature there are some restrictions to the density obtained in (31). The nature of these restrictions[8] and the explanation of why we have been successful in overcome them with our present approach may be found in the appendix.

4 Ehrenfest's Theorem

From the above relations it is possible to derive the Ehrenfest's theorem in a straightforward manner. To see this consider first the equation satisfied by the characteristic function Z_Q (3). Writing this characteristic function up to second order in the infinitesimal displacement δx we get

$$\begin{aligned} Z_Q(x, \delta x; t) = \psi^\dagger(x; t)\psi(x; t) & \left[1 + \delta x \int \psi^\dagger(x'; t) \frac{\partial \psi(x'; t)}{\partial x'} dx' + \right. \\ & \left. + \frac{(\delta x)^2}{2} \int \psi^\dagger(x'; t) \frac{\partial^2 \psi(x'; t)}{\partial x'^2} dx' \right]. \end{aligned} \quad (32)$$

Now, taking this expression into (3) and separating the zeroth and first order terms in the infinitesimal displacement δx , we get the two equations

$$-i\hbar \frac{\partial}{\partial t}(\psi^\dagger(x; t)\psi(x; t)) - \frac{\hbar^2}{m} \frac{\partial}{\partial x}(\psi^\dagger(x; t)\psi(x; t)) \int \psi^\dagger(x'; t) \frac{\partial \psi(x'; t)}{\partial x'} dx' = 0, \quad (33)$$

and

$$\begin{aligned}
& -i\hbar \frac{\partial}{\partial t} (\psi^\dagger(x;t)\psi(x;t)) \int \psi^\dagger(x';t) \frac{\partial \psi(x';t)}{\partial x'} dx' - \\
& -i\hbar (\psi^\dagger(x;t)\psi(x;t)) \int \psi^\dagger(x';t) \frac{\partial \psi(x';t)}{\partial x'} dx' - \\
& - \frac{\hbar^2}{m} \frac{\partial}{\partial x} (\psi^\dagger(x;t)\psi(x;t)) \int \psi^\dagger(x';t) \frac{\partial^2 \psi(x';t)}{\partial x'^2} dx' + \frac{\partial V(x)}{\partial x} (\psi^\dagger(x;t)\psi(x;t)) = 0.
\end{aligned} \tag{34}$$

Multiplying the first equation by x and integrating we find

$$\frac{\partial \bar{x}}{\partial t} = \bar{p}, \tag{35}$$

while, integrating the second equation in x , we find

$$\frac{\partial \bar{p}}{\partial t} = - \overline{\left(\frac{\partial V(x)}{\partial x} \right)}. \tag{36}$$

These last two equations are exactly the mathematical expressions of the Ehrenfest's theorem[7].

Moreover, with the probability density on configuration space written as $\rho(x;t) = |\psi(x;t)|^2$, the equation (33) becomes

$$\frac{\partial \rho(x;t)}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\overline{p(t)}}{m} \rho(x;t) \right] = 0, \tag{37}$$

while using this equation into (34) we get

$$\rho(x;t) \left[\frac{\partial \overline{p(t)}}{\partial t} + \frac{\partial V(x)}{\partial x} \right] + \frac{\overline{(\delta p)^2}}{m} \frac{\partial \rho(x;t)}{\partial x} = 0, \tag{38}$$

where

$$\overline{(\delta p)^2} = \overline{p(t)^2} - \overline{p(t)}^2. \tag{39}$$

These last two equations are precisely those we would get if we integrate the Liouville equation in the variable p after multiplying it by 1 and p , respectively, using a separable density function of the type given in (31)[14].

5 Application: Harmonic Oscillator

We now want to make a simple application of the previous results to the harmonic oscillator problem. Thus, we begin with the harmonic oscillator probability amplitude given by

$$\psi_n(x;t) = (\alpha^2/4)^{1/2} \left(\frac{1}{2^n n!} \right)^{1/2} e^{-\alpha^2 x^2/2} H_n(\alpha x) e^{-iE_n t/\hbar}, \tag{40}$$

where $H_n(x)$ are the Hermite polynomials and $\alpha = \sqrt{m\omega/\hbar}$, with m the mass and ω the frequency related with the problem.

Using the approach of the second section, the phase-space probability density function may be written as[9, 10, 11, 12]

$$F_n^B(x, p; t) = \frac{1}{\pi\hbar} (-1)^n \exp\left(-\frac{2H}{\hbar\omega}\right) L_n\left(\frac{4H}{\hbar\omega}\right), \quad (41)$$

which clearly assumes, for the excited states, negative values within some ranges of the variable $H = p^2/2m + m\omega^2 x^2/2$.

The phase-space probability density function of the present approach we uphold in the third section is simply

$$F_n^A(x, p; t) = |\psi_n(x; t)|^2 |\varphi_n(p; t)|^2, \quad (42)$$

where $\varphi_n(p; t)$ is the solution of the problem on the momentum space—the Fourier transform of $\psi(x; t)$.

The results obtained for the expectation values by these two phase-space probability densities are exhibited in Table I. We may see from this table that the expectations will coincide for all cases but those involving the product $x^n p^m$. While the phase-space probability density function (41) presents some correlation between the variables x and p , these correlations are absent from the one in (42). In the present harmonic oscillator case, this will affect, for example, the expectations of the energy dispersions, which become higher for the uncorrelated density.

This result was expected since, for the excited states, the densities spread more and more with respect to the variable H and its dispersion, calculated with the probability density F^B , remains constant only because this density assumes negative values within some ranges of H ; since this negativity regions do not appear in the probability density F^A , we may expect the spread in the energy to get larger, following the density spread itself. Thus, the probability density F^B presents the strange behavior of having the momentum and position dispersion getting larger and larger, while the energy dispersion being kept constant.

We have also plotted the first three phase-space probability density functions according to the prescription (42). They are given by figures I, II and III. These figures may be contrasted with those related with the prescription (41) (see, for example, Olavo[13]).

6 conclusion

In this paper we have been successful in obtaining positive definite phase-space probability density functions. Although this result does not change very deeply the formal aspects of the quantum theory, since this theory will continue to be given by the Schrödinger equation, it does constitute an important epistemological acquisition. Indeed, since the very beginning of this series we have being

arguing that quantum mechanics is a mere classical statistical mechanics, performed upon configuration space, for systems in thermodynamic equilibrium[14]. However, it will be rather inconvenient for such an epistemology to have its main symbol of liaison (the phase-space probability density) between these two levels of descriptions (phase-space and configuration space) not being capable of having a physical interpretation.

Thus, being capable of calculating the probability density functions on phase-space not suffering from the negativity problem gives us the final justification to our previous assertions. Now the theory has all its symbols with some acceptable physical interpretation attached to it and we may finally say, without fearness, that quantum mechanics is just an *ensemble* statistical theory performed upon configuration space and related with thermodynamic equilibrium situations.

The next paper will be devoted to the epistemological consequences of the formalism derived in this series of papers[6] and will constitute its natural logical closure.

A Restriction upon the Density

Cohen[8] has derived a generalized method of obtaining phase-space density functions. He was aiming at giving a general procedure of obtaining all the phase-space probability distributions proposed at the time his paper was published. He thus begins by writing this phase-space probability density function as

$$F(x, p; t; f) = \frac{1}{4\pi^2} \int \int \int e^{-i\theta x - i\tau p + i\theta u} f(\theta, \tau; t) \times \\ \times \psi^\dagger(u - \frac{1}{2}\tau\hbar; t) \psi(u + \frac{1}{2}\tau\hbar; t) d\theta d\tau du. \quad (43)$$

In this case, a wide variety of probability densities may be obtained by just fixing the function $f(\theta, \tau; t)$ (e.g. the Wigner function in (2) may be found by fixing $f = 1$).

In his conclusions he argues that: “It is commonly held that the uncertainty principle by itself precludes the possibility of the existence of a joint distribution of position and momentum. However, this is not so. For example, the choice

$$f(\theta, \tau; t) = \frac{\int \int |\psi(x)|^2 |\phi(p)|^2 e^{i\theta x + i\tau p} dx dp}{\int \psi^\dagger(u - \frac{1}{2}\tau\hbar) e^{i\theta u} \psi(u + \frac{1}{2}\tau\hbar) du} \quad (44)$$

leads to

$$F(x, p; t) = |\psi(x)|^2 |\phi(p)|^2, \quad (45)$$

which is certainly a well-defined joint distribution and from which the uncertainty principle follows in the usual manner. The reason why a true joint distribution cannot be defined is because no choice of f yields a distribution which gives the correct quantum mechanical expectation values for all observables

when calculated through phase-space integration. That is, no f exists such that, if the correspondence of quantum to classical variables (..) is

$$g(x, p) \rightarrow \hat{G}, \quad (46)$$

for some f , then also

$$H(g(x, p)) \rightarrow H(\hat{G}), \quad (47)$$

for the same f , where H is any function”.

However, this was exactly what we have obtained in the third section of this paper. Indeed, our method of operator formation was

$$\hat{x} = x \text{ and } \hat{p} = -i\hbar \frac{\partial}{\partial x'} \quad (48)$$

for which any function H will be such that

$$H(g(x, p)) \rightarrow H(\hat{G}) = H(g(\hat{x}, \hat{p})), \quad (49)$$

since \hat{x} and \hat{p} commute, denying ambiguities related with operator ordering to appear.

This is precisely the point of disagreement. When constructing his correspondence rule, Cohen[15] has explicitly assumed that the operators related with the position and the momentum are those given by $\hat{x} = x$ and $\hat{p} = -i\hbar\partial/\partial x$, which do not commute. This precludes, in his approach, the possibility of finding a function f which gives, at the same time, a specific correspondence rule, the density as in (45) (for example), and all the expectations values predicted by quantum mechanics as equal to those calculated by using the derived correspondence rule.

Thus, with the choice (43) he was able to find a function f which gives the phase-space probability density as (45). With this function f he was able to find the correspondence rule which takes a function $g(x, p)$ into its related operator \hat{G} . But when calculating the expectation values with the phase-space probability density and the operator \hat{G} , he was not able to make the results agree. Looking at our own development, we may see that the problem is due to the fact that, in his last derivation step, Cohen[15] imposes that the variables x and p be substituted by the non-commuting operators $\hat{x} = x$ and $\hat{p} = -i\hbar\partial/\partial x$, which may not be accommodated within our present approach. Indeed, it is the commuting of the operators in (48) that guarantees the validity of the expectation values calculated with the uncorrelated probability density (45), as we can see by (24), and it is obvious that we will have exactly the same results while calculating the expectations with the phase-space probability density F^A or using the operator given by (49), which is our correspondence rule.

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n	\overline{E}	$\overline{x^2}$	$(\Delta x \Delta p)$	$(\overline{x^4})$	$(\overline{x^2 p^2})_A$	$(\overline{x^2 p^2})_B$	$(\Delta E)_A^2$	$(\Delta E)_B^2$
0	1/2	1/2	1/2	3/4	1/4	1/4	1/4	1/4
1	3/2	3/2	3/2	15/4	9/4	5/4	3/4	1/4
2	5/2	5/2	5/2	39/4	25/4	13/4	7/4	1/4
3	7/2	7/2	7/2	75/4	49/4	25/4	13/4	1/4

Table 1: Some expectation values calculated with the present probability density function in phase space ($A \equiv F^A$) and the one usually found in the literature ($B \equiv F^B$). The results coincide whenever the index is absent.

TABLES

FIGURES

Figure 1 - Plot of the phase space probability density function for $N=0$.

Figure 2 - Plot of the phase space probability density function for $N=1$.

Figure 3 - Plot of the phase space probability density function for $N=2$.

FIGURE I

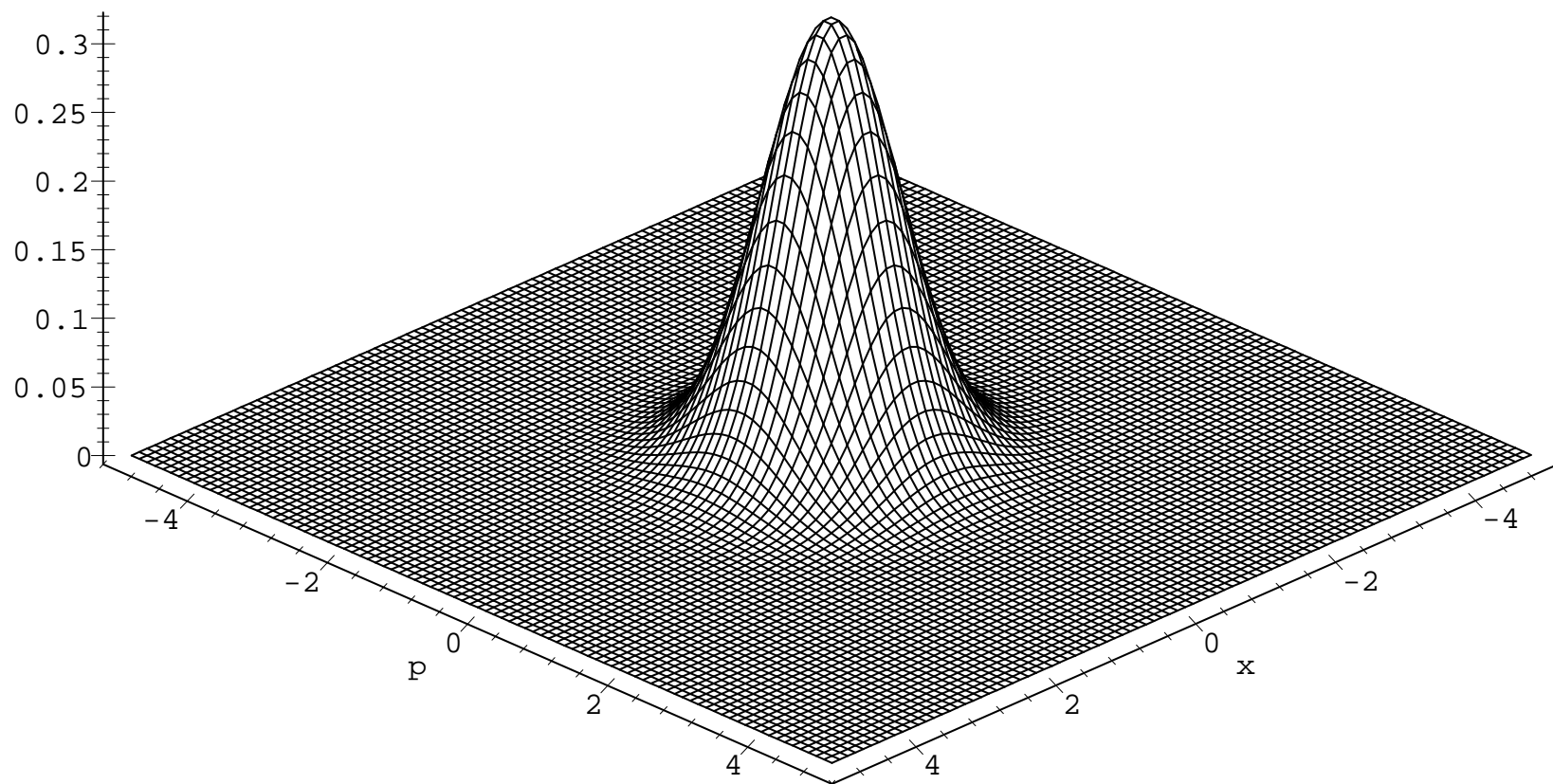


FIGURE II

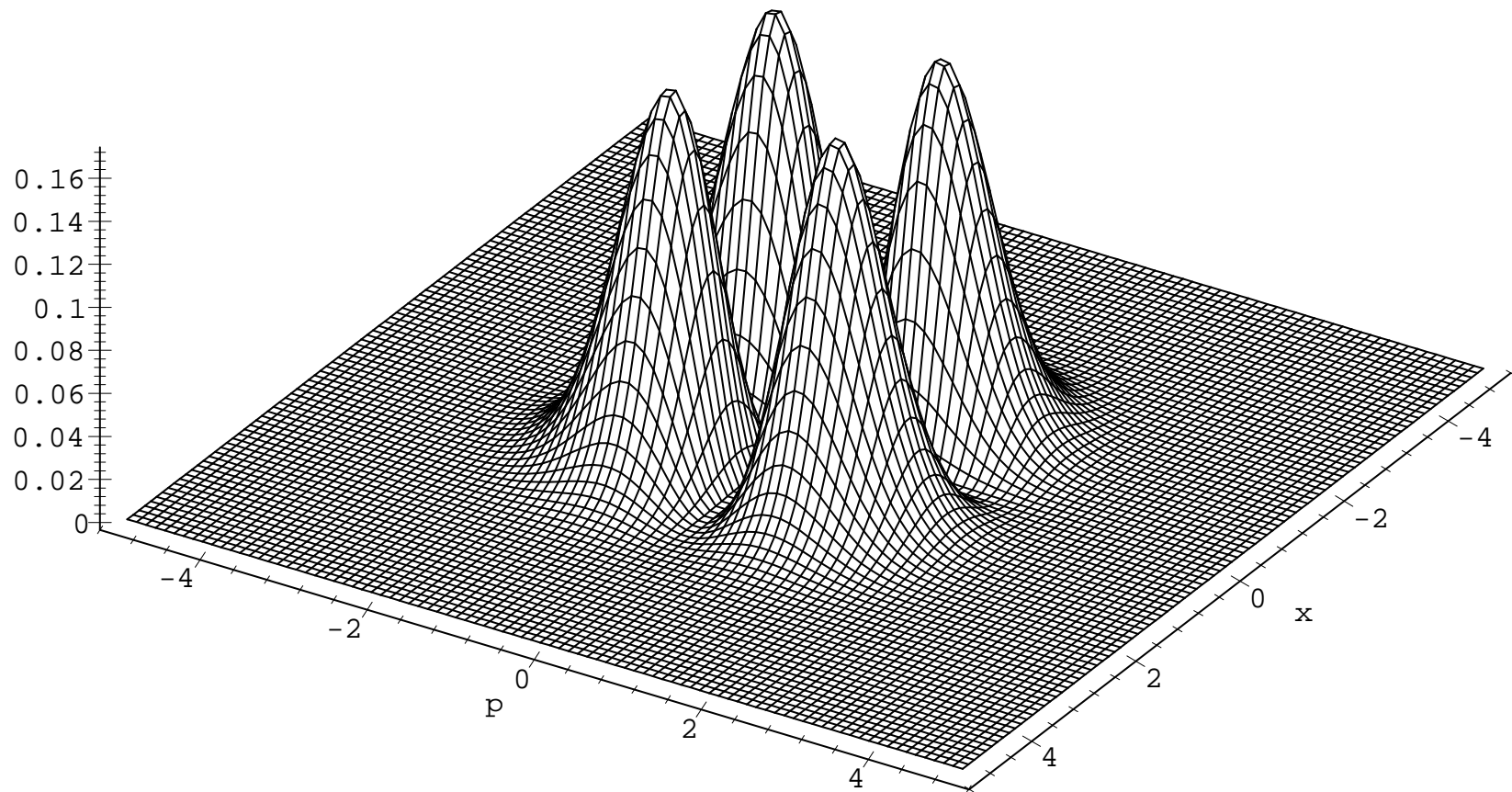


FIGURE III

